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The Development and Applications of Methods for Designing Optimal Portfolios

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Abstract. This paper explores the development and application of linear and mixed-integer programming techniques to solve complex decision-making problems in both financial portfolio optimization and modern family expenditure problems. It begins by reviewing classical financial models such as Markowitz's Mean-Variance framework, then explains its development including CAPM, APT, MAD and a series of following important models that utilize linear optimizing tools from QP, LP, ILP to the extensive MIP, MILP, MIQP and make a contrast of all of their characteristics and application contexts. By modeling household expenditure as a resource allocation problem under budgetary, risk, and utility constraints, the analysis integrates income and expenditure data from China, the U.S., and Japan, revealing policy-driven disparities in resource distribution. The integration of risk measures such as CVaR further illustrates the relevance and flexibility of linear optimization methods in complex, real-life scenarios.

Keywords: Portfolio Optimization, Linear Programming (LP), Mixed-Integer Programming (MIP), Risk Management, Modern Family Expenditure

1. Introduction

Since the mid-20th century, numerous models have been utilized to solve some problems of getting the optimal portfolios, and in these ages, most of these problems are solved by using linear programming methods. While looking through the development of methods of designing these optimal portfolios, there is a clear development that these methods are becoming more comprehensive and complex as time pass by in order to solve more situations about investments while avoiding risks under different backgrounds. The article below discusses some significant models of that and make a vertical comparison of them.

2. The development of linear optimizing tools of solving portfolio problems

The beginning of utilizing linear optimizing tools into solving optimal portfolios can be traced back to 1952, when Markowitz construct MPT (Modern Portfolio Theory) utilizing QP (quadratic programming) to maximize the discounted (or capitalized) value of future returns in addition to include an allowance of risk [1]. Or the rate at which we capitalize the returns from particular securities vary with risk. Then the model can be constructed like:

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Supposing that portfolio contains n different assets, rate of return of asset i is random with expected value m_i.

The invest function $x_i \geq 0$, while x_i represents as the investment ratio vector.

Let C denote the covariance matrix of asset returns. The objective is to minimize the portfolio risk:

$$\frac{1}{2} x^T C x$$

And expected return should be at least a minimal rate of return, which denotes as "r"

Then

$$\sum_{i=1}^{n} m_i x_i \geq r$$

With the sum of fraction x_i's should add up to a total supposed as 1:

$$\sum_{i}^{n} x_i = 1$$

Then the problem becomes a QP problem. We just need to construct an image of the function to get the optimal solution.

As the time comes to 1960s, economists started to construct the capital asset pricing model (CAPM). While the CAPM is actually a result in equilibrium of MPT. This model describes the relationship between systematic risk, or the general risks of investing, and expected return for assets, particularly in stocks. It is a finance model establishing a linear relationship between the required return rate on an investment and risk. In the model, Sharpe and Lintner, the constructors of CAPM, emphasize that assuming an individual is willing to act on the basis of only two parameters of this distribution: its expected value and standard deviation [2, 3]. Apart from the basic optimal objectives, risks and profits, CAPM also takes the market risk into consideration. And this model can be represented by a total utility function of the form:

$$E(R_i)=R_f+\beta_i(E(R_m)-R_f)$$

when $E(R_i)$ denotes the expected return on asset i;

R_f denotes the risk-free interest rate (e.g. Treasury bond rate);

$E(R_m)$ denotes the expected return on the market portfolio;

β_i denotes asset i's systematic risk coefficient relative to the market, where

$$\beta_i = \frac{Cov(R_i, R_m)}{Var(R_m)}$$

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and if β_i =1: it moves in sync with the market

if β_i >1: more sensitive than the market (high risk with high return)

if β_i <1: more stable than the market (low risk with low return)

In this model, $E(R_i)$can be expressed as a column vector, so it still can be dealt with MPT which utilize QP.

After CAPM, in 1979, a new model called APT (Arbitrage Pricing Theory), which was raised by Ross, turning to LP (or sometimes ILP) to deal with the optimal estimate [4]. The arbitrage model was proposed as an alternative to the mean variance capital asset pricing model, introduced by Sharpe, Lintner, and Treynor. However, it assumed there are no arbitrage opportunities in the market, and that asset returns can be decomposed into a linear combination of exposures to various economic factors. For example, in investment, APT can also incorporate investors' financial situations and momentum to improve trading strategies. In international market, APT can also take exchange rates and changes of global prices into measurement.

The principal relation that emerges from the mean variance model holds that for any asset, i, its expected return(R_i)can be expressed as:

$$E(R_i) = R_f + \beta_{i1}F_1 + \beta_{i2}F_2 + \cdots + \beta_{ik}F_k$$

Where R_f represents for the non-risk return; $F_k$represents for the risk premium for the k systemic risk factor; β_{ik} represents for the sensitivity of asset i to factor F_k. It is actually a linear relation.

APT itself doesn't limit factors that influence the investment, so it is an open model and derives to some multi-factor model like Fama-French three-factor model and their five-factor model while also Carhart four-factor model, which take different factors into their measurement. No matter how much factors are involved, it is actually an LP or ILP problem with the constraints of no net-investment and no factor exposure.

In 1991, MAD (Mean Absolute Deviation) model is also a practical and linear alternative to the traditional Markowitz mean-variance portfolio optimization model. It replaces variance as a measure of portfolio risk with mean absolute deviation, which is computationally simpler and leads to a Linear Programming (LP) formulation. Konno and Yamazaki, who proposed this model, demonstrate that a portfolio optimization model using the L1 risk (mean absolute deviation risk) function can remove most of the difficulties associated with the classical Markowitz's model while maintaining its advantages over equilibrium models [5]. In particular, the L1 risk model leads to a linear program instead of a quadratic program, so that a large-scale optimization problem may be solved on a real time basis. Statistics show that the L1 risk model generates a portfolio quite similar to that of the Markowitz's model within a fraction of time required to solve the latter.

The MAD formulation can be expressed:

Variables:

x_i: fraction of total capital invested in asset i

r_{it}: return of asset i in time period t

T: number of historical periods

d_t: deviation from the mean portfolio return in period t

Objective: minimize

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$$\frac{1}{T}\sum_{t=1}^{T} d_t$$

subject to:

$$d_t \geq R_t - \boldsymbol{R} \;\; d_t \geq \boldsymbol{R} - R_t \;\; where\, R_t = \sum_{i=1}^{n} r_{it} x_i$$

$$\sum_{i=1}^{n} \mathbf{r}_i x_i \geq R_{min}$$

$$\sum_{i=1}^{n} x_i = 1$$

$$x_i \geq 0$$

In the late 20th century, the concept of VAR has become popular in the economic world. VaR estimates the potential loss of a set of investments (with a given probability) under normal market conditions over a specified time period. For instance, if a portfolio of stocks has a one-day 5% VaR of $1 million, that means that there is a 0.05 probability that the portfolio will fall in value by $1 million or more over a one-day period if there is no trading. Informally, a loss of $1 million or more on this portfolio is expected on 1 day out of 20 days. So according to this definition, when a loss random variable L is given, then the VaR at level α is the α-quantile of the loss distribution.

$$\mathrm{VaR}_{\alpha}\big(\mathrm{L}\big) = \inf\left\{\mathrm{l}\epsilon\mathbf{R}\mathrm{P}(\mathrm{L} \leq \mathrm{l}) \geq \alpha\right\}$$

This means that the probability that the loss does not exceed VaR is at least α.

Then in the 21st century, the implementing of CVaR as a more stable and optimized measure of risk, can be transformed into a linear programming (LP also sometimes ILP) problem by introducing context variables and auxiliary variables. It is based on VaR, as the risk measure estimates the maximum potential loss in the value of an investment or portfolio over a given time period. As VaR cannot be optimized or modeled into linear program. Then the constructing of CVaR is able to solve this flaw of VaR.

Minimizing CVaR of a portfolio is closely related to minimizing VaR, as it has been already observed from the definition of these measures. The CVaR at level α is the expected loss given that the loss exceeds VaR, as the average of losses in the worst (1−α)% scenarios. Then in the paper of

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Rockafellar and Uryasev about CVaR, they give a specific formula that implementing linear program to measure CVaR [6]:

Let f (x; y) be the loss associated with the decision vector x, to be chosen from a certain subset X of R^n, and the random vector y in R^m. The vector x can be interpreted as representing a portfolio, and the vector y stands for return vectors at different uncertainties. In the formula, q represents for the total number of return scenarios, and β represents for the confidence level such as 0.95,0.99.

$$F_\beta\left(x,\alpha\right)=\alpha+\frac{1}{q(1-\beta)}\sum_{i=1}^{q}\left[f(x,y_i)-\alpha\right]$$

The expression above is what we need to minimize, while subject to:

$$f(x,y_i)-\alpha\geq 0 \qquad i=1,2,\cdots,q$$

$$\mathrm{x}\geq 0 \quad 1^{\mathrm{T}}\mathrm{x}=1$$

Then the question changes to an LP problem that can handle real-world constraints better than QP utilized for variance-based models.

3. The characteristics and applications of different optimal tools

There are also other papers that contribute to more complicated linear programs utilized in portfolio problems. In some specific occasions such as asset ceiling limit, industry constraint and transaction costs, MIP, MIQP or MILP are involved.

In 2000, Chang, Meade, Beasley, Sharaiha extend the standard model to include cardinality constraints that limit a portfolio to have a specified number of assets, and to impose limits on the proportion of the portfolio held in a given asset [7]. After realizing the weaknesses of Markowitz's QP model of not being sustainable and practical for real investors, they incorporate discrete investment decisions using Mixed-Integer Programming (MIP). It demonstrates how integer programming tools solve portfolio selection problems under complex constraints.

In 2007, a paper by Mansini, Ogryczak, Speranza demonstrates that the MAD model, originally formulated as a Linear Programming (LP) problem when only linear constraints (including the budget constraint) are present, remains efficiently solvable as an LP problem even when a single additional quadratic constraint is introduced [8]. This quadratic constraint often represents a limit on portfolio variance or tracking error variance relative to a benchmark. Then this model is actually an MIQP model. It can also be useful to reflect real-world regulatory and practical investment conditions and provides a computationally tractable framework to solve complex, constraint-rich optimization problems. The paper itself is complicated that use professional calculus calculating, but it can also be simplified to a MIQP problem.

In 2011, Woodside-Oriakhi, Lucas & Beasley have integrated MIP in practical fund management either. Though the model they constructed is truly an MIQP model, the heuristic approximations used in their paper can be treated as an MIP model practically [9]. As MIP usually is much easier in

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computation, it can solve problems under cardinality constraints, minimum transaction units or lot sizes and minimum budget constraints. They point out that the model is also a QMIP, but provided that the number of assets for which we have to make a decision as to whether they are in or out of the portfolio is small, it can be solved relatively quickly to proven optimality. That is why it can be treated as an MIP or hybrid MIP/LP problem.

Above all, there are some methods of linear programs-from QP to LP and then more modern and far more advanced model like MIP, MIQP and MILP. Therefore, let's can learn from these different linear programs of their different characteristics and the specific problems they can solve.

Table 1. The characteristics and cases for different linear optimized tools

Tool	Characteristics	Use Cases
QP	-Quadratic objective, linear constraints -Convex, solvable efficiently -Directly model variance(risk)	Markowitz mean-variance optimization
LP	-Linear objective -Continuous objective -Fastest to solve, widely applicable -Usually not being utilized alone	- Basic portfolio selection - Resource allocation - Transportation and logistics optimization - Production planning
ILP/IP	- Discrete decision modeling - Often combinatorial - Slow to solve at scale	- Cardinality-constrained portfolios - Project selection - Facility location problems
MIP	- Mix of - Flexible and realistic - Balances complexity and tractability	- Cardinality-constrained portfolios - Investment limits - Lot-size Constraints
MIQP	- Quadratic objective + integer variables - Combines risk modeling and discrete controls - Computationally expensive	- Advanced portfolios with variance constraints - Regulated fund allocation - Sector limit portfolios
MILP	- Linear objective and integer variables. - Handles complex constraints (real world rules) - Easier to solve than MIQP	- Trading models - Sector-balanced portfolios - Sector limits, min/max holdings

According to these methods and thoughts, though there are many contemporary economic problems are solved by heuristic and metaheuristic methods, deep learning and other non-linear programming methods, we can still utilize these seemly traditional methods into problems in reality.

4. A specific utilization of linear optimizing tools of family resources distribution problems

Like the thought of risk and revenue in portfolio, family expenses are also facing the condition of allocating limited resources in multiple objectives. Nguyen Huu A., Mbodji, O., Nguyen-Huu, A., & Pirvu, T. A. have raised an approach based on optimal sharing rule [10], which the rule tells us that in investment process, investors utility preferences, weights and market conditions decide how much they consume and shoulder the risks. This study pointed out that the optimal expending structure is not only limited by the economic conditions of a family but also affected by the rights of negotiation and shapes of preference function. By this thought, the following part is going to see different projects where families need to expend as "multiple assets". A household-level expenditure portfolio optimization model is constructed by weighted utility function and risk penalty term, so as to maximize the overall satisfaction of households under the constraint of total budget.

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In modern lives, the increasing cost of living is laying pressure on families, while these families are also seeking for a higher life quality. These families' limited assets are expended into different areas including basic living cost (food, drink, clothes, traffic, houses), education cost, medical cost, leisure cost (including trip, toys, cost to a fancy park or other services to enjoy themselves), and also some risk investment and savings. How to allocate them seems to be an important and practical problems.

Then we refer these living cost, education cost, medical cost, leisure cost, investment and savings as project i=1,2,3,4,5,6...n and consider a rather long time period such as a whole year. The families' income I decide the maximum of the amount of how much they can spend in all these fields. Meanwhile, each of the project poses different significance, as living cost and medical cost cannot be replaced but leisure cost can be cut down. However, the significance can also be affected by different family's preference, and the rights of these projects denotes for w. Some of these projects also include risks, like investment and medical cost, while some of them contains a big elasticity like leisure and investment. Then if a family spends x_i on different project, then the satisfactory

$$u_i = \frac{(x_i + 1)^{e_i} - 1}{e_i}$$

Where elasticity e<1, then increasing x_i cannot bring significant satisfactory like cost on medical and education. Where elasticity e>1, then increasing x_i can bring significant satisfactory like cost on leisure and investment. The living cost's e is approximately approaching 1.

So what the problem needs to solve is to allocate x in different projects:

$$\max \sum_{i=1}^{n} w_i u_i, \text{ where } \sum_{i=1}^{n} x_i \leq I$$

Then the fact that different families have their different preference, like some families who don't have children or are unwilling to make investment. And even each member of these families has different perspectives over consumption and risk investment. The feasibility could be affected by their rights. If the preference and risks are being considered, then it transfers to:

$$\max \left[\sum_{i=1}^{n} w_i u_i - \sum_{i=1}^{n} \lambda r_i x_i \right], \text{ where } \lambda \text{ denotes for the risk aversion index}$$

Then this is a MILP problem where the first term denotes for the effects of investment, and the second term denotes for the corresponding punishment of each project.

And according to the National Bureau of Statistics household consumption expenditure report, in 2024, Chinese individual in average can possess 41,314CNY. And the chart below shows that the $w_i$for family's average valuing these projects.

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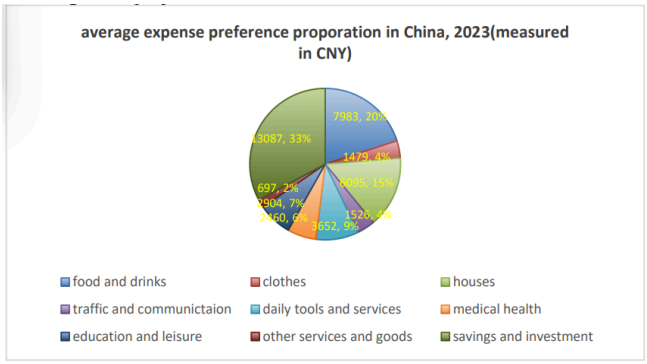

Figure 1. The annual average expenditure proportion of a Chinese in 2023

As the data above don't contain the difference between the urban and rural, while also some of the difference like education and leisure are mixed up. So, the chart above can only be used for reference, and also different families have their own preference.

Then the following part is going to take an insight in US and Japan, where the situation is shown below.

The annual individual average income is $40,559 in 2023, data from US Bureau of Labor Statistics.

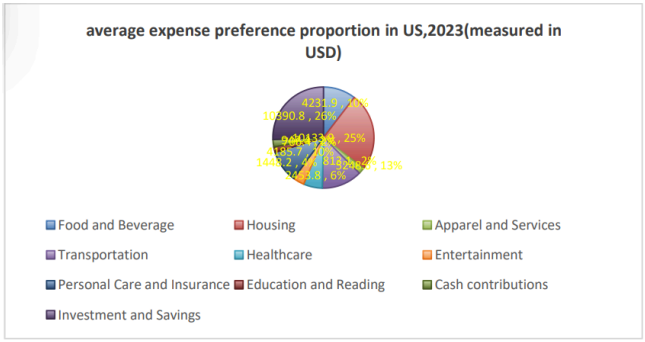

Figure 2. The annual average expenditure proportion of an American in 2023

And in 2023, the annual individual average income is $16,906, data from Statistics Bureau of Japan

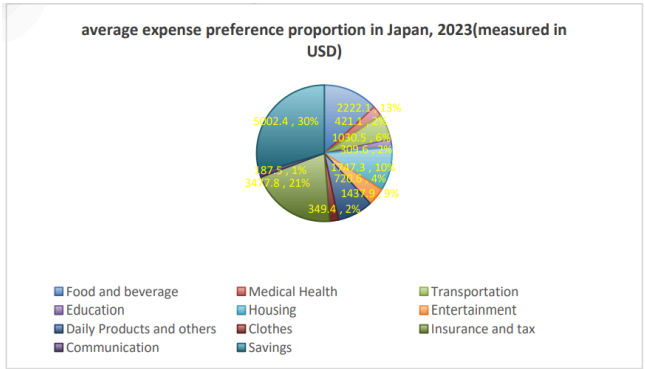

Figure 3. The annual average expenditure proportion of a Japanese in 2023

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So there are differences between Japan, China, and America. Though the total disposable income is different, the distribution of that can still be comparable. It is obviously large for Chinese's proportion on food and beverage, and American's proportion on housing is obviously large than other two countries. From the diagram shown above, we can also notice that the insurance and tax occupy a large proportion for Japan. This may be caused by the measurement of different countries, as China and America are mixing up tax in different projects. However, about savings, each of these countries have taken up about 30% in expenditure. Then estimations for living cost(including food, clothes, houses, communication and transportation) occupying around 50% and 60% where the housing always occupying the biggest proportion, medical health occupying around 10%(including insurance), education occupying around 5%, entertainment occupying less than 4%, other cost including tax, charity, services and the deviation that the estimation makes occupying less than 15%, then the last is the asset that can be used in investment and savings. So for a decision maker in a family, each of these distributions is around these rates. The projects above can be divided into minor projects(like living cost divided into houses, food, clothes and so on) while calculating in reality.

Then consider the elasticity. According to the reality, experience and some existed study showing that in developed countries [11, 12]:

Table 2. Data of the elasticity in some aspects concluded by experience and analysis

Project	Elasticity	Project	Elasticity	Project	Elasticity
Food	0.3-0.6	Medical Health	0.2-0.7	Traffic	1.1(estimated)
Education	1.0-1.5	Houses	<1(around 0.8)	Entertainment	>1
Clothes	0.6-1.0	Daily Products	0.3-0.5	Insurance	0.5-0.8

The elasticity data are also average rate. For example, each of different sorts of food have their own elasticity, as grain and wheat are smaller than 0.2 and beef and seafood are above 1, while also eating out is something with higher elasticity. The data also may not fit in other countries like China, where education in suburban and rural area is regarded as rigid demand as the policy while traffic and housing in cities can be regarded as luxury [13].

The subsequent part proceeds to examine another parameter, r, the risk or uncertainty of different investment projects. It can be embodied in minor aspects including the uncertainty in demands, prices, unpredictable and uncontrollable spending pressure and volatility of investment or consumption consequences. The methods are shown below:

Table 3. Some suggested methods being implemented to calculate the risk of different investment projects

Method	Description	Example
Statistical volatility	Using the standard deviation of expenditure divided by the mean	$\sigma_i = \frac{std(x_i)}{mean(x_i)} = CV$
Subjective assignment	According to the policy sensitivity and event sensitivity scores	Medical: about 0.7 Food: about 0.2
Multinational research method	Using OECD/ EU/ Japan/ NTO/ China data extrapolated	Education in China is lack of elasticity with a rate about 0.3 while full of elasticity in US with a rate about 0.6

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For subjective assignment, it is easily imagined that the food and clothes are something with low uncertainty level, while medical health, and traffic are something with middle uncertainty level that they are dependent on policies, the debasing on prices and development of technology. Additionally, the uncertainty of education is related to the ages of children and their intelligence or learning ability. Entertainment and investment are something with high uncertainty level that usually like gambling and possibility games. Therefore, in calculation, these r can be assigned in different specific constant. It also can be calculated by using historic data and getting the standard deviation.

However, in reality, occasions like medical accident, education risk and emergent events including firing and unemployment may happen. Those things could strongly decrease the satisfactory which may be calculated with a higher possibility rather than normal distribution, causing the distribution of that in a left fat-tail distribution. Therefore, a function can be defined to calculate the expenditure in excess of distribution, and the CVaR constraints that mentioned above can better solve these events. Then the extreme risk can be expressed as:

$$\sum_i \lambda_{2i} CVaR_\alpha(L_i)$$

where the λ_{2i} can also denote as the risk aversion index in this circumstance.

In reality, the $CVaR_{95\%(usually)}$ can be calculated by comparing and analyzing investment like CHFS to simulate the distribution and getting an estimated rate of CVaR. For example, in medical area, China, Japan and America have the rate about 25%, 15%, 40%, respectively. The difference is caused by the strong reliance on subsidy in China, the universal health coverage in Japan, and the inequality distribution in commercial insurance in America.

Therefore, the above model framework can be concluded as:

$$\max\left[\sum_{i=1}^{n} w_i \frac{(x_i+1)^{e_i}-1}{e_i} - \sum_{i=1}^{n} \lambda_1 r_i x_i - \sum_i \lambda_{2i} CVaR_\alpha(L_i)\right]$$

It bridges economic theory with real-life expenditure allocation, and demonstrates how classical portfolio optimization methods can be extended beyond financial markets into the field of household consumption decision-making.

However, there remains significant potential to extend this framework by the incorporation of other analytical tools and methodologies. For instance, stochastic programming, modeling income volatility or price shocks via scenario trees, could be integrated to capture the uncertainty of future income streams or the probabilistic nature of unexpected expenditures, offering more resilient solutions. Robust optimization provides an alternative approach for dealing with data ambiguity without assuming probability distributions, which may be useful in modeling household decision-making under incomplete information or behavioral bias. Moreover, with the increasing availability of large-scale consumer and expenditure datasets, machine learning models—such as decision trees, clustering algorithms, fuzzy logic and reinforcement learning—can be employed to learn patterns of household behavior and preferences, and to generate adaptive allocation policies that respond to changes in individual preference shifts, life stage(including child's growth and the extension or

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departure of a family), or macroeconomic factors(such as debasing of currency, disasters or economic depression).

Future study can focus on the further extension of expenditure in families. Besides of utilizing these methodologies mentioned above to solve more complicated problems, some more specific formula and model could also be used to test data and reflect economic problems. Moreover, more accurate parameters like the elasticity and risk parameter can also be accessed in different regions and circumstances. The model may not only construct for deciders in a family to allocate expend in different projects to get each dollar valuable and useful, but it can also be extended for the government to take some acts to regulate economic activities and giving some welfare and subsidy when necessary.

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