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Distance-Edge-Monitoring Sets in Cube-Connected Cycles Networks

Binhong Gu

Yangming College, Ningbo University, Ningbo, China
2503402020@qq.com

Abstract. Distance-edge-monitoring set is a graph-theoretic model for detecting and locating edge failures in networks by means of distance measurements. In this paper, we study the distance-edge-monitoring set of the s-dimensional cube-connected cycles network $CCC(s)$. By analyzing the structural constraints of shortest paths in $CCC(s)$, we construct an explicit distance-edge-monitoring set of size 2^{s-1}. Consequently, we obtain the upper bound $\mathrm{dem}(CCC(s)) \leq 2^{s-1},\ s \geq 2$. The proof is carried out by considering cube edges and cycle edges separately and showing that every edge is contained in all shortest paths between a suitable probe vertex and a target vertex. The result provides a compact and regular monitoring strategy for cube-connected cycles networks and may serve as a basis for further study of exact values and related monitoring problems on interconnection networks.

Keywords: cube-connected cycles, distance-edge-monitoring set, graph theory, network monitoring, interconnection networks

1. Introduction

Network monitoring is a fundamental problem in distributed systems and communication networks. In recent years, graph-theoretic monitoring models have received increasing attention. A distance-edge-monitoring set [1] is a set of probe vertices that can detect edge failures through changes in distances to other vertices.

The cube-connected cycles (CCC) network is an important interconnection topology that combines the high connectivity of hypercubes with the simplicity of cycles. This is constructed by substituting every vertex of a hypercube with a cycle. More precisely, the s-dimensional cube-connected cycles network $CCC(s)$ has $s \cdot 2^s$ vertices, and every vertex has degree 3.

This paper aims to investigate the distance-edge-monitoring number of $CCC(s)$. A previous proof attempt toward the exact value $\mathrm{dem}(CCC(s)) = 2^{s-1}$ relied on a projection argument to the hypercube, but that lower-bound argument is not yet complete. To ensure mathematical correctness, we establish here a rigorous upper bound by constructing an explicit monitoring set of size 2^{s-1}. Therefore, the key finding of this work is

$$\mathrm{dem}(CCC(s)) \leq 2^{s-1}, \qquad s \geq 2. \tag{1}$$

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2. Preliminaries

In this work, every graph is finite, undirected, and simple. For any graph H, we denote its edge set by $E(H)$ and its vertex set by $V(H)$. The order of H is $|V(H)|$, and the number of edges is denoted by $|E(H)|$.

Let $X \subseteq V(H)$. Then $H[X]$ denotes the subgraph of H induced by X. Given any edge subset $X \subseteq E(H)$, we write $H - X$ for the graph obtained by deleting all edges in X from H. If $X = e$, we simply denote it by $H - e$.

For $v \in V(H)$, the neighborhood of v in H is $N_H(v)$, and its degree is $\deg_H(v) = |N_H(v)|$. The maximum and minimum degrees of H are written as $\Delta(H)$ and $\delta(H)$, respectively.

We denote by P_n the path on n vertices, by C_n the cycle on n vertices, by K_n the complete graph on n vertices, and by $K_{m,n}$ the complete bipartite graph with bipartition sizes m and n.

For any $x, y \in V(H)$, let $d_H(x, y)$ denote their distance in H, i.e., the length of a shortest x–y path in H.

For more details, please refer to [2].

2.1. Distance-edge-monitoring sets

Definition 1. [1] Let $H = (V, E)$ be a connected graph. For a subset $M \subseteq V$ and an edge $e \in E$, define

$$P(M, e) = \{(u, v) \mid u \in M,\ v \in V,\ d_H(u, v) \neq d_{H-e}(u, v)\}. \tag{2}$$

If $P(M, e) \neq \varnothing$ for any edge $e \in E$, then M is called a *distance-edge-monitoring set* of H. The minimum cardinality of a distance-edge-monitoring set of H is called the *distance-edge-monitoring number* of H, denoted by $\mathrm{dem}(H)$.

Alternatively, an edge e is monitored by M if there exists a pair $u \in M$ and $v \in V(H)$ such that e belongs to all shortest u–v paths.

For more details, please refer to [3, 4].

2.2. Hypercubes

Let Q_n denote the n-dimensional hypercube [5]. Its vertex set is

$$V(Q_n) = \{0, 1\}^n, \tag{3}$$

that is, all binary strings of length n. Two vertices are adjacent precisely when they differ in exactly one coordinate.

The following facts are well known:

(1) $|V(Q_n)| = 2^n$;
(2) $|E(Q_n)| = n2^{n-1}$;
(3) Q_n is n-regular;
(4) $\mathrm{diam}(Q_n) = n$;
(5) Q_n is bipartite;
(6) Q_n is edge-transitive and vertex-transitive.

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2.3. Hamming distance

Given two binary strings $y = (y_1, \ldots, y_n)$ and $x = (x_1, \ldots, x_n)$ of length n, their Hamming distance [6] is

$$\mathrm{dist}_H(y, x) = \sum_{i=1}^{n} \delta(y_i, x_i), \tag{4}$$

where $\delta(y_i, x_i) = 1$ if $y_i \neq x_i$, and $\delta(y_i, x_i) = 0$ otherwise.

In the hypercube Q_n, graph distance coincides with Hamming distance:

$$d_{Q_n}(u, v) = \mathrm{dist}_H(u, v). \tag{5}$$

2.4. Cube-connected cycles networks

The s-dimensional cube-connected cycles network [7], denoted by $CCC(s)$, is constructed as follows:

(1) Begin with the s-dimensional hypercube Q_s;
(2) Replace every vertex of Q_s by an s-cycle;
(3) For each dimension-i hypercube edge, connect it to the i-th position on the corresponding cycle.

Thus, every vertex of $CCC(s)$ can be represented as an ordered pair

$$\langle w, i \rangle, \tag{6}$$

where $w \in \{0, 1\}^s$ identifies a hypercube vertex and $i \in \{0, 1, \ldots, s-1\}$ identifies a position on the cycle.

Two vertices $\langle w, i \rangle$ and $\langle w', i' \rangle$ are adjacent precisely when one of the following holds:

(1) $i' \equiv i \pm 1 \pmod{s}$ and $w = w'$;
(2) w' differs from w only in the i-th bit and $i = i'$.

For $i, j \in \{0, 1, \ldots, s-1\}$, define the distance on the cycle by

$$d_{\mathrm{cyc}}(i, j) = \min\{s - |i - j|, |i - j|\}. \tag{7}$$

For more details, please refer to [8-10].

Lemma 1. *Consider two vertices $u = \langle w_1, i_1 \rangle$ and $v = \langle w_2, i_2 \rangle$ in $CCC(s)$. Let*

$$H = \mathrm{dist}_H(w_1, w_2), \tag{8}$$

and let $K = \{k_1, \ldots, k_H\}$ be the set of coordinates in which w_1 and w_2 differ. Then:

(1) every u–v path must visit all cycle positions corresponding to coordinates in K;
(2) the length of any shortest u–v path equals the minimum total cycle-movement length needed to start at i_1, visit all positions in K, and end at i_2, plus the fixed H bit-flip steps.

Proof. In $CCC(s)$, the k-th bit can be flipped only at cycle position k. Therefore, any path from u to v must visit every required position in K. Each bit flip contributes exactly one step, and all remaining steps are cycle moves. Hence, a shortest path is obtained by minimizing the total cycle-movement length subject to visiting all required positions, and then adding the fixed number H of bit-flip steps.

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3. Main result

In this section, we construct a distance-edge-monitoring set of cardinality 2^{s-1} for $CCC(s)$.

Theorem 1. *For every integer $s \geq 2$,*

$$\operatorname{dem}(CCC(s)) \leq 2^{s-1}. \tag{9}$$

More precisely, the set

$$M = \{\langle w, 0\rangle \mid w \in \{0,1\}^s, \text{ the } 0\text{-th bit of } w \text{ is } 0\} \tag{10}$$

is a distance-edge-monitoring set of $CCC(s)$.

Proof. Since the 0-th bit is fixed to be 0 while the remaining $s-1$ bits are arbitrary, we have

$$|M| = 2^{s-1}. \tag{11}$$

It remains to prove that every edge of $CCC(s)$ is monitored by M.

Case 1. e is a cube edge.

Suppose

$$e = \langle w, i\rangle\langle w', i\rangle, \tag{12}$$

where w differs from w' only in the i-th bit.

Subcase 1.1. $i = 0$.

Then w' and w differ only in the 0-th bit, so exactly one of them has 0-th bit equal to 0. Choose the corresponding endpoint as $x \in M$, and let the other endpoint be y. Since y and x are adjacent, the unique shortest path from y to x is precisely e. Hence e is monitored.

Subcase 1.2. $i \neq 0$ *and the* 0*-th bit of* w *is* 0.

Then $\langle w, 0\rangle \in M$. Let

$$x = \langle w, 0\rangle, \qquad y = \langle w', i\rangle. \tag{13}$$

To go from x to y, one must flip the i-th bit exactly once, and this can only be done at cycle position i. Therefore each shortest path from y to x must first move along the cycle from position 0 to position i, and then traverse the cube edge e to reach y. Hence e lies on each shortest path from y to x.

Subcase 1.3. $i \neq 0$ *and the* 0*-th bit of* w *is* 1.

Define $w^{(0)}$ as the binary string resulting from flipping the 0-th bit of w. Then the 0-th bit of $w^{(0)}$ is 0, so

$$x = \langle w^{(0)}, 0\rangle \in M. \tag{14}$$

Let

$$y = \langle w', i\rangle. \tag{15}$$

To reach y from x, both the 0-th and the i-th bits must be flipped. If the 0-th bit is flipped first, then one obtains a path of length

$$1 + 1 + d_{\text{cyc}}(0, i). \tag{16}$$

If the i-th bit is flipped first, then one must still return to position 0 to flip the 0-th bit and then go back toward the target configuration, which requires at least two additional cycle traversals of length $d_{\text{cyc}}(0, i) \geq 1$. Hence such a route is strictly longer. Therefore every shortest path must flip the 0-th bit first and then traverse the cube edge e to flip the i-th bit. So e is monitored.

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Case 2. e is a cycle edge.

Suppose

$$e = \langle w, i+1\rangle\langle w, i\rangle, \tag{17}$$

where the index $i+1$ is taken modulo s.

Subcase 2.1. $i = 0$ *and the* 0*-th bit of* w *is* 0.

Take

$$x = \langle w, 0\rangle \in M, \qquad y = \langle w, 1\rangle. \tag{18}$$

These two vertices are adjacent, so the unique shortest path from y to x is exactly e.

Subcase 2.2. $i = 0$ *and the* 0*-th bit of* w *is* 1.

Let $w^{(0)}$ be obtained from w by flipping the 0-th bit. Then

$$x = \langle w^{(0)}, 0\rangle \in M, \qquad y = \langle w, 1\rangle. \tag{19}$$

Every path from y to x must first flip the 0-th bit at position 0, reaching $\langle w, 0\rangle$, and then move one step along the cycle to $\langle w, 1\rangle$. This last step is precisely the edge e, and any alternative route is longer. Hence e is monitored.

Subcase 2.3. $i \neq 0$ *and the* 0*-th bit of* w *is* 0.

Let $w^{(i)}$ be obtained from w by flipping the i-th bit. Since $i \neq 0$, the 0-th bit of $w^{(i)}$ remains 0, so

$$x = \langle w^{(i)}, 0\rangle \in M, \qquad y = \langle w, i+1\rangle. \tag{20}$$

To go from x to y, one must flip the i-th bit at position i. Hence every shortest path must reach $\langle w, i\rangle$ and then take the final cycle step to $\langle w, i+1\rangle$. Therefore, e is present on every shortest route between y and x.

Subcase 2.4. $i = s-1$ *and the* 0*-th bit of* w *is* 1.

Here $i + 1 \equiv 0 \pmod{s}$, so

$$e = \langle w, s-1\rangle\langle w, 0\rangle. \tag{21}$$

Take

$$x = \langle w^{(0)}, 0\rangle \in M, \qquad y = \langle w, s-1\rangle. \tag{22}$$

Any shortest path must first flip the 0-th bit at position 0, reaching $\langle w, 0\rangle$, and then move one step along the cycle to $\langle w, s-1\rangle$. This step is exactly e, so e is monitored.

Subcase 2.5. $i \neq 0$, $i \neq s-1$, *and the* 0*-th bit of* w *is* 1.

Let $w^{(0,i)}$ be the binary string obtained from w by flipping both the 0-th and the i-th bits. Then its 0-th bit is 0, so

$$x = \langle w^{(0,i)}, 0\rangle \in M, \qquad y = \langle w, i+1\rangle. \tag{23}$$

To reach y from x, the 0-th and i-th bits must both be flipped.

If the 0-th bit is flipped first, then one gets a path of length

$$1 + d_{\text{cyc}}(0, i) + 1 + 1 = d_{\text{cyc}}(0, i) + 3. \tag{24}$$

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If the i-th bit is flipped first, then one must still return to position 0 to flip the 0-th bit and then move to position $i+1$, yielding a path of length at least

$$d_{\mathrm{cyc}}(0,i)+1+d_{\mathrm{cyc}}(i,0)+1+d_{\mathrm{cyc}}(0,i+1)=2d_{\mathrm{cyc}}(0,i)+d_{\mathrm{cyc}}(0,i+1)+2. \tag{25}$$

Since $i\neq 0$ and $i+1\neq 0$, we have

$$d_{\mathrm{cyc}}(0,i)\geq 1, \qquad d_{\mathrm{cyc}}(0,i+1)\geq 1, \tag{26}$$

and therefore

$$2d_{\mathrm{cyc}}(0,i)+d_{\mathrm{cyc}}(0,i+1)+2>d_{\mathrm{cyc}}(0,i)+3. \tag{27}$$

Hence every shortest path must flip the 0-th bit first and then the i-th bit, after which the final step to the target is exactly the cycle edge e. Thus e is monitored.

Combining all cases, M monitors every edge of $CCC(s)$. Hence M is a distance-edge-monitoring set of $CCC(s)$, and

$$\mathrm{dem}(CCC(s))\leq |M|=2^{s-1}. \tag{28}$$

4. Conclusion

In this paper, we studied the distance-edge-monitoring problem for cube-connected cycles networks. By analyzing the shortest-path structure of $CCC(s)$ and constructing a specific family of probe vertices, we proved that

$$\mathrm{dem}(CCC(s))\leq 2^{s-1}, \qquad s\geq 2. \tag{29}$$

The construction is uniform, which makes it potentially useful for monitoring design in structured interconnection networks.

Future work may proceed in several directions. First, it is natural to seek a rigorous lower bound matching the upper bound obtained here, in order to determine the exact value of $\mathrm{dem}(CCC(s))$. Second, weighted variants and stronger monitoring models may also be considered.

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