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Modeling and Validation of MDOF Structural Vibration and Penetration Dynamics: From Analytical Derivations to Finite Element Simulations

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Abstract. This paper presents a comprehensive study of multi-degree-of-freedom (MDOF) structural vibration and projectile–target penetration dynamics using analytical, numerical, and simulation-based approaches. For the undamped three-degree-of-freedom mass–spring system, equations of motion were derived and expressed in matrix form, enabling the determination of natural frequencies, mode shapes, and unknown mass parameters. This analysis verified the model's validity and demonstrated the fundamental characteristics of MDOF vibration systems. In parallel, a reduced-order projectile–target model was developed, coupling rigid-body motion with axial vibration through equivalent stiffness and damping. Analytical derivations, MATLAB simulations, and Python-based nonlinear modeling highlighted the roles of damping, elastic compression, and resistance forces in penetration dynamics. To validate these findings, finite element simulations using ANSYS/Autodyn were performed, capturing projectile deformation, energy dissipation, and penetration depth. The results collectively confirm that simplified models, when cross-validated with high-fidelity simulations, can effectively capture the essential physics of vibration and impact, offering practical insights for engineering design, vibration control, and protective structure development.

Keywords: MDOF vibration system, equations of motion, natural frequencies, mode shapes, projectile penetration, equivalent stiffness and damping, nonlinear resistance, finite element simulation, ANSYS, structural dynamics

1. Introduction

The vibration of engineering structures is a widespread phenomenon, and applications are used to either amplify or reduce the structural vibrations. In fact, most human activities are related to vibration or its derivatives [1]. For example, the construction of a bridge considers not only the amount of weight it can carry but also its ability to combat vibrations, and buildings are also designed to offset the vibration from earthquakes. Through studies of vibration, people can explain

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phenomena in daily life, invent objects that work with vibrations, and give early warning of some natural disasters such as earthquakes and volcanic eruptions. Vibrations of Multiple Degrees of Freedom (MDOF) extend beyond the Single Degree of Freedom (SDOF) circumstance by accounting for the additional modes of independent motion, such as bending and twisting, which occur more often in real life.

Significant studies have been carried out on the dynamic behavior of multi-degree-of-freedom (MDOF) structural systems. Wenjie Guo and colleagues employed the finite element method (FEM) to investigate vibrations in conical–cylindrical–spherical shell structures, proposing a new linear formulation approach [2]. Meanwhile, Bor-Tsuen Wang and his team performed modal analysis on MDOF systems using free vibration response data, relying solely on finite difference techniques [3]. These prior works predominantly utilized advanced computational software and complex analytical frameworks. In contrast, this study adopts straightforward numerical techniques and leverages widely-used simulation tools such as ANSYS to develop two simplified models. These models are designed to replicate distinct MDOF configurations, enabling the analysis of free vibration responses in a discrete lumped mass-string system, as well as impact-induced vibrations in a simulated projectile-target interaction.

Two distinct models were developed, one of which represents a three-degree-of-freedom mass-spring system without damping, operating on a frictionless surface and potentially influenced by external forces. Based on the system's configuration, equations describing the displacement of each mass over time are derived, enabling the determination of the time-dependent relative displacement between the second and first masses. To facilitate analysis, the masses are treated as point masses located at the center of each respective mass element [4-8].

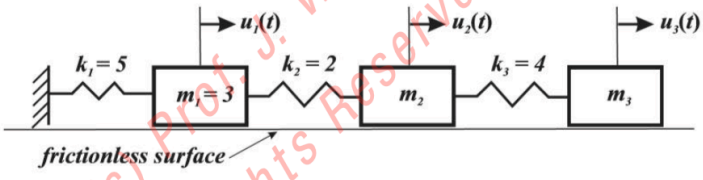
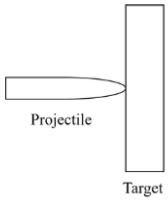


Figure 1. Undamped mass-spring system

Another system involves a single projectile penetrating a target, experiencing dynamic forces during the impact process. Based on the system's configuration, equations describing the penetration depth of the projectile as a function of time are established. From these, the relative penetration velocity with respect to the target over time can be derived. Additionally, for analytical simplification, both the projectile and the target are modeled as point masses located at their respective geometric centers.



Schematic Diagram of Projectile Penetration Process

Figure 2. Projectile impact system

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2. Problem

2.1. Problem background

This part focuses on the response of a multiple-degree-of-freedom, undamped, discrete vibration system. The objective of this part is to analyze the dynamic response of the mechanical system comprising three interconnected masses (m_1, m_2, and m_3 as shown below), deriving and solving the equations of motion to understand its vibrational behavior. This involves determining the natural frequencies, mode shapes, and magnitudes of the masses under specified conditions, enabling the validation of the system's dynamic model through simulation.

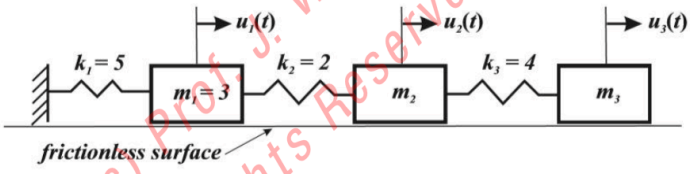


Figure 3. During the vibration process, no external force acts on the system, and the system is undamped

2.2. Problem description

In this problem, the system consists of three masses (m_1, m_2, m_3) interconnected by springs with stiffness coefficients (k_1, k_2, k_3) on a frictionless surface with m_1 =3 and all stiffness coefficients given. The masses are denoted as m_1 = 3 kip-sec^2/in, with m_2 and m_3 unspecified, and their displacements are represented by $u_1(t)$, $u_2(t)$, and $u_3(t)$, respectively. The springs have stiffness values of k_1 = 5 kip/in connecting m_1 to a fixed wall, k_2 = 2 kip/in between m_1 and m_2, and k_3 = 4 kip/in between m_2 and m_3. This configuration forms a 3-degree-of-freedom (DOF) system, where the motion of each mass is influenced by the spring forces, and the frictionless surface ensures no energy dissipation caused by the friction.

2.3. Prompt

Consider a system of three masses that are interconnected by springs, where the stiffnesses ki are illustrated in the diagram. The lumped masses are denoted by m_1, m_2, and m_3. The units of the stiffness coefficients k_{ij} are kip/in. And that for m's is kip-sec^2/in. The U.S./British customary units can be converted to the SI units.

(a) Create a free-body diagram of each mass and write its equation of motion. Then describe the equations of motion in terms of the matrix form.

(b) A mode of vibration with the largest component set to unity has the form:

$$\emptyset_1 = \begin{Bmatrix} 0.44444 \\ -0.66667 \\ 1.00000 \end{Bmatrix} \text{ (not mass orthonormalized)}$$

If m_1 = 3, determine the magnitudes of the masses m_2 and m_3.

(c) Find the three natural frequencies of this system.

Note that the problem used U.S./British customary units kip/inch and kip-sec^2/inch, which will be converted to SI units after we have obtained the ultimate answer for each question

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2.4. Problem-solving process

Part (a):

In this question, we first need to draw the free-body diagram and determine the equation of motion for each mass. Hence, the first step to approaching this problem is to analyze the forces acting on each of the masses. We will neglect the analysis of the gravitational force and the normal force since these two forces are equal and cancel out, and do not affect the vibration action and the motion of each mass.

Mass m1 is connected to a fixed wall on the left via spring k_1 = 5 kip/in and to mass m2 on the right via spring k_2 = 2 kip/in. Mass m2 is positioned between m1 and m3, connected via spring k_2 = 2 kip/in on the left and k_3 = 4 kip/in on the right. Mass m3 is the rightmost mass, connected only to m2 via spring k_3 = 4 kip/in (free end on the right). Therefore, we can draw the free-body diagrams for the individual masses (Figure 4).

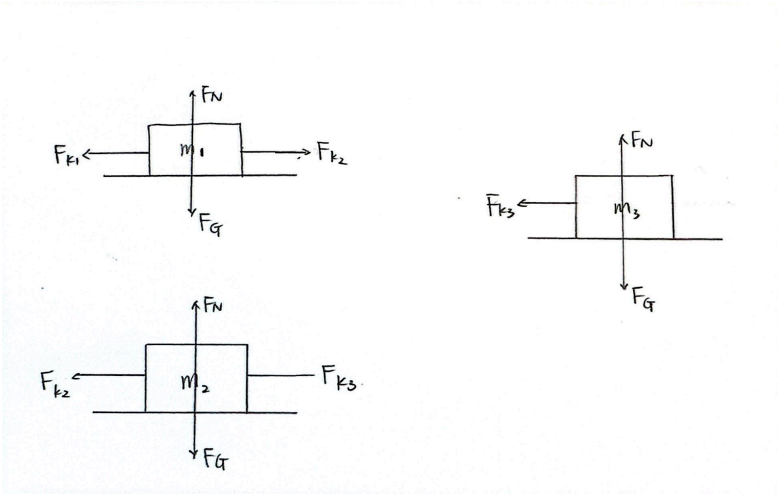


Figure 4. Free-body diagrams

For the forces acting on mass m1, the force from spring k1 exerts a restoring force proportional to the displacement u1 from the equilibrium position. If u1 > 0, the spring is stretched, pulling m1 back to the left, resulting in a force of $-k_1u_1 = -5u_1$ kip (negative direction). The force from spring k_2 depends on the relative displacement between m1 and m_2. If $u_2 > u_1$, the spring is stretched, pulling m_1 to the right, resulting in a force of $+k_2 (u_2 - u_1) = +2 (u_2 - u_1)$ kip. Conversely, if $u_2 < u_1$, the spring is compressed, pushing m1 to the left, which is still captured by the same expression. Therefore, we can obtain the net force. By Newton's Second Law of Motion, the net force equals $m_1\ddot{u}_1$:

$$m_1\ddot{u}_1 = -k_1u_1 + k_2(u_2 - u_1) \tag{1}$$

Likewise, by applying the same analysis method to mass 2 and mass 3 and Newton's Second Law of Motion, we can obtain their equation of motion as well:

$$\begin{cases} m_2\ddot{u}_2 = -k_2 (u_2 - u_1) + k_3 (u_3 - u_2) \\ m_3\ddot{u}_3 = -k_3 (u_3 - u_2) \end{cases} \tag{2}$$

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Hence,

$$\begin{cases} m_1\ddot{u}_1 = -k_1u_1 + k_2\left(u_2 - u_1\right) \\ m_2\ddot{u}_2 = -k_2\left(u_2 - u_1\right) + k_3\left(u_3 - u_2\right) \\ m_3\ddot{u}_3 = -k_3\left(u_3 - u_2\right) \end{cases} \tag{3}$$

Where we can also write:

$$\begin{cases} m_1\ddot{u}_1 + (k_1 + k_2)u_1 - k_2u_2 = 0 \\ m_2\ddot{u}_2 - k_2u_1 + (k_2 + k_3)u_2 - k_3u_3 = 0 \\ m_3\ddot{u}_3 - k_3u_2 + k_3u_3 = 0 \end{cases} \tag{4}$$

Then, we can transform the equations into matrix form by separating the mass matrix and the stiffness coefficient matrix,

$$\begin{aligned} \boldsymbol{M} &= \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \\ \boldsymbol{k} &= \begin{bmatrix} k_1 + k_2, & -k_2, & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} = \begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -4 \\ 0 & -4 & 4 \end{bmatrix} \end{aligned} \tag{5}$$

Using $M\ddot{u} + ku = 0$ and by plugging in the values from the equation of motion, we can obtain:

$$\begin{matrix} 3 & 0 & 0 \\ [0 & m_2 & 0 \\ 0 & 0 & m_3 \end{matrix} \begin{matrix} \ddot{u}1 \\ ]\left[\ddot{u}2\right] \\ \ddot{u}3 \end{matrix} + \begin{matrix} 7 & -2 & 0 \\ [-2 & 6 & -4 \\ 0 & -4 & 4 \end{matrix} \begin{matrix} u1 \\ ]\left[u2\right] \\ u3 \end{matrix} = 0. \tag{6}$$

Part (b):

In this question, we are asked to obtain the magnitudes of the masses m2 and m3. We're also given a vibration mode shape with the largest component set to 1, written as $\phi_1 = \begin{pmatrix} 0.44444 \\ -0.66667 \\ 1.00000 \end{pmatrix}$ (not adjusted for mass normalization), and we know m1=3 kip-sec²/in. we have the specific stiffness values: k1=5 kip/in, k2=2 kip/in, and k3=4 kip/in. Our goal is to figure out the sizes of masses m2 and m3.

To find these masses, we use the principle that the mode shape fits into the equation for free vibration, which looks $[K]\phi_1 = \omega^2\,[M]\phi_1$. Here, $\phi_1 = [0.44444, -0.66667, 1.00000]^T$ is the mode shape, and ω^2 is the square of the natural frequency we need to determine. Since the mode isn't normalized for mass, we plug it into the equation and work out the masses, assuming we know the stiffness matrix [K] from part (a).

The stiffness matrix [K] from part (a) is:

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$$\left[K\right]=\begin{bmatrix}k_1+k_2 & -k_2 & 0\\ -k_2 & k_2+k_3 & -k_3\\ 0 & -k_3 & k_3\end{bmatrix} \quad . (7)$$

The mass matrix [M] is diagonal with $m_1 = 3$, m_2 and m_3 to be determined:

$$\left[M\right]=\begin{bmatrix}m_1 & 0 & 0\\ 0 & m_2 & 0\\ 0 & 0 & m_3\end{bmatrix}=\begin{bmatrix}3 & 0 & 0\\ 0 & m_2 & 0\\ 0 & 0 & m_3\end{bmatrix} \quad (8)$$

Now, apply the eigenvalue equation $[K]\phi_1 = \omega^2 [M]\phi_1$.

On the left side of the eigenvalue equation, by plugging in the stiffness coefficient values, we can get:

$$\left[\boldsymbol{K}\right]\phi_1 \approx \begin{pmatrix}4.44442\\ -8.88890\\ 6.66668\end{pmatrix} \quad (9)$$

Likewise, on the right side of the eigenvalue equation, by plugging in the given vibration mode shape into the mass matrix, we can get:

$$\omega^2\left[\boldsymbol{M}\right]\phi_1 = \begin{pmatrix}3\cdot\omega^2\cdot 0.44444\\ m_2\cdot\omega^2\cdot\left(-0.66667\right)\\ m_3\cdot\omega^2\cdot 1.00000\end{pmatrix} \quad (10)$$

Since the question already gives us that m1 = 3, we can use this value to obtain that $\omega^2 \approx 3.3333$. Plugging this into both rows, we can get that

$$m_2 \approx 4.0kip - \frac{sec^2}{in} \ and \ m_3 \approx 2.0kip - \frac{sec^2}{in} \quad (11)$$

Part (c):

The objective of part (c) is to determine the three natural frequencies of the 3-degree-of-freedom (DOF) system depicted in the provided diagram and described by the given equations of motion.

Formulation of Equations of Motion:

The process begins with the establishment of the equations of motion for the three masses based on Newton's second law.

$$\begin{cases}m_1\ddot{u}_1 = -k_1u_1 + k_2\left(u_2 - u_1\right)\\ m_2\ddot{u}_2 = -k_2\left(u_2 - u_1\right) + k_3\left(u_3 - u_2\right)\\ m_3\ddot{u}_3 = -k_3\left(u_3 - u_2\right)\end{cases} \quad (12)$$

Matrix Representation:

We convert the scalar equations into a matrix form, defining the mass matrix [M] and stiffness matrix [K]:

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$$[M] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} [K] = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \tag{13}$$

Substituting all the given values from the previous questions and the prompt yields:

$$M = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}, K = \begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -4 \\ 0 & -4 & 4 \end{bmatrix} \tag{14}$$

This step sets the stage for applying the eigenvalue problem to find the natural frequencies.

Eigenvalue Problem Setup:

We assumed a harmonic solution of the form $u(t) = \phi e^{i\omega t}$, where ϕ is the mode shape vector and ω is the natural frequency. Substituting this into the equation of motion $M\ddot{u} + Ku = 0$ (noting $\ddot{u} = -\omega^2 u$)results in:

$$[K]\phi_1 = \omega^2 [M]\phi_1 \tag{15}$$

or equivalently,

$$(K - \omega^2 M)\phi = 0 \tag{16}$$

For non-trivial solutions, the determinant of $(K - \omega^2 M)$ must be zero:

$$det(K - \omega^2 M) = 0 \tag{17}$$

This equation will yield a cubic characteristic equation in ω2 \omega^2 ω2, from which the three natural frequencies can be extracted.

Computation of the Characteristic Equation:

Construct the matrix $K - \omega^2 M$ by subtracting $\omega^2 M$ from [K]:

$$K - \omega^2 M = \begin{bmatrix} 7 - 3\omega^2 & -2 & 0 \\ -2 & 6 - 4\omega^2 & -4 \\ 0 & -4 & 4 - 2\omega^2 \end{bmatrix} \tag{18}$$

The determinant of this 3x3 matrix is calculated using the rule for a 3x3 determinant:

$$det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a\left(ei - fh \right) - b\left(di - fg \right) + c\left(dh - eg \right) \tag{19}$$

Applying this to $K - \omega^2 M$:

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$$det(K-\omega^2 M = (7-3\omega^2)\left[(6-4\omega^2)(4-2\omega^2)-(-4)(-4)\right]-(-2)\left[(-2)(-2\omega^2)-0\right]+0$$

Simplifying the terms step-by-step:
First term: $(6-4\omega^2)(4-2\omega^2)-16=24-12\omega^2-16\omega^2+8\omega^4-16=8\omega^4-28\omega^2+8$
Second term: $-(-2)(4-2\omega^2)=8-4\omega^2$
Thus, the determinant becomes:

$$(7-3\omega^2)(8\omega^4-28\omega^2+8)-(-2)(8-4\omega^2) \tag{20}$$

The planner expands this expression, noting the handwritten expansion:

$$(7-3\omega^2)(8\omega^4-28\omega^2+8)+2(8-4\omega^2) \tag{21}$$

Distributing and combining like terms:

$$7\cdot 8\omega^4=56\omega^4 \tag{22}$$

$$7\cdot(-28\omega^2)=-196\omega^2 \tag{23}$$

7·8=56

$$-3\omega^2\cdot 8\omega^4=-24\omega^6 \tag{24}$$

$$-3\omega^2\cdot(-28\omega^2)=84\omega^4 \tag{25}$$

$$-3\omega^2\cdot 8=-24\omega^2 \tag{26}$$

2·8=16

$$2\cdot(-4\omega^2)=-8\omega^2 \tag{27}$$

Combining together:

$$-24\omega^6+(56+84)\omega^4+(-196-24-8)\omega^2+56+16=-24\omega^6+140\omega^4-228\omega^2+72 \tag{28}$$

Simplification and Solving the Cubic Equation:
Simplified down to one single equation:

$$6\omega^6-35\omega^4+53\omega^2-10=0 \tag{29}$$

To solve, substitute $\lambda=\omega^2$, transforming the equation into a cubic:

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$$6\lambda^3 - 35\lambda^2 + 53\lambda - 10 = 0 \tag{30}$$

The roots of this cubic equation are sought, with the planner testing possible rational roots (e.g., factors of 10/6) and using the handwritten values $\omega_1^2 \approx 3.333, \omega_2^2 \approx 2.28, \omega_3^2 \approx 0.22$.

Verification and Finalization:

To verify these roots, we can substitute the frequency values back into the characteristic equation or using numerical methods (e.g., Newton-Raphson) if needed. The natural frequencies are then calculated as $\omega = \sqrt{\lambda}$:

$$\omega_1 \approx \sqrt{3.333} \approx 1.826 \text{ rad/s} \tag{31}$$

$$\omega_2 \approx \sqrt{2.28} \approx 1.51 \text{ rad/s} \tag{32}$$

$$\omega_3 \approx \sqrt{0.22} \approx 0.469 \text{ rad/s} \tag{33}$$

3. Conclusion

This study combined analytical modeling, numerical simulation, and finite element analysis to investigate structural vibration and projectile–target penetration. For the three-degree-of-freedom undamped mass–spring system, equations of motion were derived and expressed in matrix form, leading to the determination of natural frequencies, mode shapes, and unknown masses. This confirmed the model's validity and illustrated fundamental properties of multi-degree vibration systems.

For the projectile–target penetration model, rigid-body motion was coupled with axial vibration through equivalent stiffness and damping. Simulations demonstrated how elastic compression, damping, and resistance forces govern the transient dynamics of the projectile head and tail. Incorporating velocity- and temperature-dependent resistance improved physical realism, while ANSYS/Autodyn simulations qualitatively supported the reduced-order results.

Overall, the research shows that simplified models, when carefully formulated and validated against higher-fidelity simulations, can effectively capture the essential physics of vibration and impact phenomena. The framework provides practical insights for vibration control, penetration resistance, and early-stage engineering design.

4. Evaluation

The proposed models effectively simplify complex mechanical systems while preserving essential physical characteristics, making them suitable for preliminary design and feasibility analysis. The combination of analytical derivation and computational simulation ensured result validation and enhanced model credibility. However, several limitations remain: the undamped MDOF model does not account for energy dissipation in real-world structures, and the projectile-target model assumes uniform material properties and neglects temperature-dependent degradation under extreme conditions. Furthermore, the assumption of constant or simplified resistance forces in certain simulations reduces the accuracy in predicting penetration depth and residual velocity, particularly for heterogeneous or layered targets. Future work should integrate advanced damping models, nonlinear material constitutive laws, and experimentally calibrated resistance functions to improve

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predictive accuracy. Despite these limitations, the study provides a valuable framework for understanding vibration and impact phenomena, offering practical guidance for engineering applications such as structural vibration control and protective material design.

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All the authors contributed equally to this work and should be considered as co-first author.

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